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The Deformation Gradient in Curvilinear Coordinates

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Abstract

This short article offers an overview of the deformation gradient and its determinant in the case of curvilinear coordinates. In particular, the relation to a pure Cartesian description and the representation relative to a curvilinear basis is treated. This brief exercise is known, but hard to be found in the basic literature on Continuum Mechanics. Further, most of the literature on constitutive modeling refers to the symbol J, but J is not generally equivalent to $\det \mathbf{F}$. The difference is derived in detail.

1 Introduction

The basic problem arises from the fact that the Jacobian of the deformation gradient is not identical to the determinant of the deformation gradient in the case of curvilinear coordinates. In order to show this, we recall some relations provided by Marsden and Hughes (1983).

$$\begin{array}{ccc} (X^1, X^2, X^3) & \xrightarrow{\Phi^i_{\mathrm{R}}(X^1,X^2,X^3,t)} & (x^1, x^2, x^3) \\ \hat{X}^i(\Theta^1, \Theta^2, \Theta^3)\Big\uparrow & & \Big\uparrow \hat{x}^i(\vartheta^1, \vartheta^2, \vartheta^3) \\ (\Theta^1, \Theta^2, \Theta^3) & \xrightarrow{\chi^i_{\mathrm{R}}(\Theta^1,\Theta^2,\Theta^3,t)} & (\vartheta^1, \vartheta^2, \vartheta^3) \end{array} \tag{1}$$

represents the occurring coordinate transformations. (X^1, X^2, X^3) and (x^1, x^2, x^3) are Cartesian coordinates in the reference and the current configuration, respectively. Accordingly, $x^i = \Phi^i_{\mathrm{R}}(X^1, X^2, X^3, t)$ defines the motion of a particle (X^1, X^2, X^3) if Cartesian coordinates are chosen.

In the reference configuration, it is possible to choose curvilinear coordinates $(\Theta^1, \Theta^2, \Theta^3)$ as well. $X^i = \hat{X}^i(\Theta^1, \Theta^2, \Theta^3)$ defines the coordinate transformation between the curvilinear and the Cartesian coordinates in the reference configuration. In the current configuration other curvilinear coordinates might be chosen, e.g. $(\vartheta^1, \vartheta^2, \vartheta^3)$, which are given by the coordinate transformation between the curvilinear and the Cartesian coordinates in the current configuration $x^i = \hat{x}^i(\vartheta^1, \vartheta^2, \vartheta^3)$. Thus, the motion can also be written by $\vartheta^i = \chi^i_{\mathrm{R}}(\Theta^1, \Theta^2, \Theta^3, t)$.

In the following, tangent and gradient vectors are required. They are defined as follows:

$$\vec{g}_i = \frac{\partial \hat{x}^k}{\partial \vartheta^i}\vec{e}_k, \qquad \vec{g}^{\,i} = \frac{\partial \hat{\vartheta}^i}{\partial x^k}\vec{e}_k \tag{2}$$

$$\vec{G}_i = \frac{\partial \hat{X}^k}{\partial \Theta^i}\vec{e}_k, \qquad \vec{G}^{\,i} = \frac{\partial \hat{\Theta}^i}{\partial X^k}\vec{e}_k. \tag{3}$$

$\vec{e}_k$ are the basis vectors in a Cartesian coordinate system. Furthermore, co- and contravariant metric coefficients relative to the reference and current configuration must be defined:

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j = \frac{\partial \hat{x}^k}{\partial \vartheta^i}\frac{\partial \hat{x}^m}{\partial \vartheta^j}\delta_{km} = \frac{\partial \hat{x}^k}{\partial \vartheta^i}\frac{\partial \hat{x}^k}{\partial \vartheta^j} \tag{4}$$

$$g^{ij} = \vec{g}^{\,i} \cdot \vec{g}^{\,j} = \frac{\partial \hat{\vartheta}^i}{\partial x^k}\frac{\partial \hat{\vartheta}^j}{\partial x^m}\delta_{km} = \frac{\partial \hat{\vartheta}^i}{\partial x^k}\frac{\partial \hat{\vartheta}^j}{\partial x^k} \tag{5}$$

$$G_{ij} = \vec{G}_i \cdot \vec{G}_j = \frac{\partial \hat{X}^k}{\partial \Theta^i}\frac{\partial \hat{X}^m}{\partial \Theta^j}\delta_{km} = \frac{\partial \hat{X}^k}{\partial \Theta^i}\frac{\partial \hat{X}^k}{\partial \Theta^j} \tag{6}$$

$$G^{ij} = \vec{G}^{\,i} \cdot \vec{G}^{\,j} = \frac{\partial \hat{\Theta}^i}{\partial X^k}\frac{\partial \hat{\Theta}^j}{\partial X^m}\delta_{km} = \frac{\partial \hat{\Theta}^i}{\partial X^k}\frac{\partial \hat{\Theta}^j}{\partial X^k} \tag{7}$$

Without a recapitulation of the theory of tensor calculus, the following properties are stated,

$$[g^{ij}] = [g_{ij}]^{-1} \quad \text{and} \quad [G^{ij}] = [G_{ij}]^{-1}, \tag{8}$$

see (Malvern, 1969; Ogden, 1984). In other words, the inverse of the matrix of covariant metric coefficients is identical to the matrix of contravariant metric coefficients.

2 Representation of the deformation gradient

The deformation gradient $\mathbf{F}(\vec{X}, t) = \operatorname{Grad} \vec{\chi}_{\mathrm{R}}(\vec{X}, t)$ is defined as the gradient of the motion of a particle $\vec{X}$ occupying the place $\vec{x}$ at time t. In curvilinear coordinates, this reads

$$\mathbf{F}(\vec{X}, t) = \frac{\partial \chi^i_{\mathrm{R}}}{\partial \Theta^k}\vec{g}_i \otimes \vec{G}^{\,k}. \tag{9}$$

If we insert the tangent vector $\vec{g}_i$ and the gradient vector $\vec{G}^{\,k}$ by the definitions $(2)_1$ and $(3)_2$, this yields

$$\mathbf{F}(\vec{X}, t) = \frac{\partial \hat{x}^m}{\partial \vartheta^i}\frac{\partial \chi^i_{\mathrm{R}}}{\partial \Theta^k}\frac{\partial \hat{\Theta}^k}{\partial X^n}\vec{e}_m \otimes \vec{e}_n. \tag{10}$$

Obviously, this represents the chain-rule

$$\frac{\partial \Phi_{\mathrm{R}}^m}{\partial X^n} = \frac{\partial \hat{x}^m}{\partial \vartheta^i} \frac{\partial \chi_{\mathrm{R}}^i}{\partial \Theta^k} \frac{\partial \hat{\Theta}^k}{\partial X^n}, \tag{11}$$

which implies the equivalence of (9) with the Cartesian representation of the deformation gradient

$$\mathbf{F}(\vec{X}, t) = \frac{\partial \Phi_{\mathrm{R}}^m}{\partial X^n} \vec{e}_m \otimes \vec{e}_n. \tag{12}$$

3 Jacobian and determinant of deformation gradient

In the following, it will be shown that the Jacobian

$$J = \det \left[\frac{\partial \chi_{\mathrm{R}}^i}{\partial \Theta^k} \right] \tag{13}$$

is not equivalent to the determinant of the deformation gradient, $\det \mathbf{F}$. This can be shown by insering relation $\vec{G}^k = G^{kl} \vec{G}_l$ as well as the Eqns.$(2)_1$ and $(3)_1$ into the deformation gradient in curvilinear representation (9), which leads to

$$\mathbf{F}(\vec{X}, t) = \frac{\partial \chi_{\mathrm{R}}^i}{\partial \Theta^k} \vec{g}_i \otimes \vec{G}^k = \frac{\partial \chi_{\mathrm{R}}^i}{\partial \Theta^k} \frac{\partial \hat{x}^m}{\partial \vartheta^i} \frac{\partial \hat{X}^n}{\partial \Theta^l} G^{lk} \vec{e}_m \otimes \vec{e}_n. \tag{14}$$

The determinant of the matrix of covariant metric coefficients (4) reads

$$\det[g_{ki}] = \left(\det \left[\frac{\partial \hat{x}^k}{\partial \vartheta^i} \right] \right) \left(\det \left[\frac{\partial \hat{x}^k}{\partial \vartheta^j} \right] \right) = \left(\det \left[\frac{\partial \hat{x}^k}{\partial \vartheta^i} \right] \right)^2 \tag{15}$$

where the multiplication theorem of matrices, $\det(\mathbf{AB}) = (\det \mathbf{A})(\det \mathbf{B})$, is exploited. Or, to put it differently,

$$\det \left[\frac{\partial \hat{x}^k}{\partial \vartheta^i} \right] = \sqrt{\det[g_{ki}]}, \tag{16}$$

which holds also for the metric coefficients in the reference configuration

$$\det \left[\frac{\partial \hat{X}^n}{\partial \Theta^l} \right] = \sqrt{\det[G_{ln}]}. \tag{17}$$

Accordingly, the determinant of the deformation gradient reads

$$\det \mathbf{F} = \underbrace{\left(\det \left[\frac{\partial \chi_{\mathrm{R}}^i}{\partial \Theta^k} \right] \right)}_{J} \underbrace{\left(\det \left[\frac{\partial \hat{x}^m}{\partial \vartheta^i} \right] \right)}_{\sqrt{\det[g_{mi}]}} \underbrace{\left(\det \left[\frac{\partial \hat{X}^n}{\partial \Theta^l} \right] \right)}_{\sqrt{\det[G_{ln}]}} \underbrace{\left(\det[G^{lk}] \right)}_{1/\det[G_{lk}]} = \tag{18}$$

$$= J \frac{\sqrt{\det[g_{mi}]}}{\sqrt{\det[G_{lk}]}}, \tag{19}$$

see property (8) as well. In other words, $\det \mathbf{F} = J$ holds only if the determinants of the covariant metric coefficients g_{ij} and G_{ij} are identical. This holds for Cartesian coordinates, but even for the most common cylindrical or spherical coordinates this is violated.

In particular, J is used in the development of constitutive models to indicate the volumetric behavior of the material, which is misleading due to the aforementioned property. Its real application are in analytical solutions for compressible materials, see (Ogden, 1984), or in beam and shell theories, where curvilinear coordinates and compressible constitutive equations are used. Thus, J as an abbreviation, instead of $\det \mathbf{F}$, has to be carefully applied.

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