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Comparison of the multiplicative decompositions $\mathbf{F} = \mathbf{F}_\Theta \mathbf{F}_\mathrm{M}$ and $\mathbf{F} = \mathbf{F}_\mathrm{M} \mathbf{F}_\Theta$ in finite strain thermo-elasticity

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Abstract

In this work out the multiplicative decomposition of the deformation gradient into a thermal and a mechanical part is investigated on the basis of a model of thermo-elasticity. The proposed multiplicative decomposition is studied to be in both order, i.e. in the form $\mathbf{F} = \mathbf{F}_\Theta \mathbf{F}_\mathrm{M}$ and $\mathbf{F} = \mathbf{F}_\mathrm{M} \mathbf{F}_\Theta$. It is shown that for the case of isotropy and the assumption of a pure volumetric temperature evolution both formulations yield the same stress state. However, in the intermediate configurations different results occur. Furthermore, some basic investigations of uniaxial tensile/compression tests with constant temperature or problems for classical strain-energies are treated.

1 Introduction

The incorporation of thermal effects into mechanically subjected deformation processes can be modeled by the multiplicative decomposition of the deformation gradient $\mathbf{F}(\vec{X},t) = \operatorname{Grad}\vec{\chi}_{\mathrm{R}}(\vec{X},t)$, into a mechanical part $\mathbf{F}_{\mathrm{M}}$ and a thermal part $\mathbf{F}_{\Theta}$, respectively,

$$\mathbf{F} = \mathbf{F}_{\mathrm{M}}\mathbf{F}_{\Theta}, \tag{1}$$

where $\vec{x} = \vec{\chi}_{\mathrm{R}}(\vec{X},t)$ defines the motion of particle $\vec{X}$ occupying the place $\vec{x}$ at time t. This proposal goes back to [Lu and Pister, 1975]. Another decomposition makes use of the reverse order

$$\mathbf{F} = \mathbf{F}_{\Theta}\mathbf{F}_{\mathrm{M}}, \tag{2}$$

see, for example, [Yu et al., 1997] and [Miehe, 1988]. However, for both decompositions there is no systematic comparison available to show the resulting strain and stress measures in view of the concept of dual variables, see [Haupt and Tsakmakis, 1989, Haupt and Tsakmakis, 1996].

The multiplicative decomposition of the deformation gradient into two parts comes from the field of plasticity, where the deformation gradient decomposes into an elastic and a plastic part, $\mathbf{F} = \mathbf{F}_{\mathrm{e}}\mathbf{F}_{\mathrm{p}}$, see [Lee and Liu, 1967, Lee, 1969]. For a double product, i.e. the multiplicative decomposition is carried out several times in order to assign various physical causes to kinematical quantities, see [Lion, 2000a, Tsakmakis and Willuweit, 2004] and its numerical treatment in [Hartmann et al., 2008] and the literature cited therein. The decomposition is also extended to constitutive models of viscoelasticity, see [Lubliner, 1985, Lion, 1997], see [Hartmann, 2002] for further literature. A further possibility makes use of the deformation gradient's decomposition into volume-preserving and volume-changing parts going back to [Flory, 1961]. If a strain-energy function is built up of two terms, one containing the volume-changing and the other the volume-preserving part, the stress state results in a pure hydrostatic stress-state caused only by the volume change and a deviatoric part only influenced by the strain-energy of the volume-preserving deformation, see, for example, [Miehe, 1994, Hartmann and Neff, 2003] and the literature cited therein.

In this short study a model of finite strain thermo-elasticity is developed making use of the property that most elastomers show nearly incompressible behavior so that the basic elasticity relation should take this into account. Following the ideas in [Hartmann and Neff, 2003], the Flory-type decomposition into an isochoric and a volumetric part for thermo-hyperelasticity is applied as well, which has the advantage of a systematic assignment of volumetric effects to the spherical part of the Cauchy-stress tensor and the volume-preserving deformation to the deviatoric stress state.

The investigations are structured as follows. First of all, the decompositions $\mathbf{F} = \mathbf{F}_{\Theta}\mathbf{F}_{\mathrm{M}}$ and $\mathbf{F} = \mathbf{F}_{\mathrm{M}}\mathbf{F}_{\Theta}$ are studied in view of the quantities in the concerning intermediate configurations. Afterwards, some brief investigations on uniaxial tensile tests are addressed for a model applicable in rubber elasticity. The underlying investigations are, in subsequent investigations on anisotropic and inelastic constitutive models, a basic investigation.

2 Strain and stress measures of decomposition $\mathbf{F} = \mathbf{F}_{\Theta}\mathbf{F}_{\mathbf{M}}$

According to the sketch in Fig. 1 the multiplicative decomposition (2) is applied,

$$\mathbf{F} = \mathbf{F}_{\Theta}\mathbf{F}_{\mathrm{M}} = \mathbf{F}_{\Theta}\hat{\mathbf{F}}_{\mathrm{M}}\overline{\mathbf{F}}_{\mathrm{M}}, \tag{3}$$

where the mechanical part

$$\mathbf{F}_{\mathrm{M}} = \hat{\mathbf{F}}_{\mathrm{M}}\overline{\mathbf{F}}_{\mathrm{M}} \tag{4}$$

is decomposed into a volume-preserving part $\overline{\mathbf{F}}_{\mathrm{M}}$ and a volume-changing part $\hat{\mathbf{F}}_{\mathrm{M}}$,

$$\hat{\mathbf{F}}_{\mathrm{M}} = (\det\mathbf{F}_{\mathrm{M}})^{1/3}\mathbf{I}, \tag{5}$$
$$\overline{\mathbf{F}}_{\mathrm{M}} = (\det\mathbf{F}_{\mathrm{M}})^{-1/3}\mathbf{F}_{\mathrm{M}}. \tag{6}$$

Frequently, the thermal part is assumed to be purely volumetric Al

$$\mathbf{F}_{\Theta} = \varphi^{1/3}\mathbf{I}, \qquad \varphi = \hat{\varphi}(\Theta - \Theta_0), \tag{7}$$

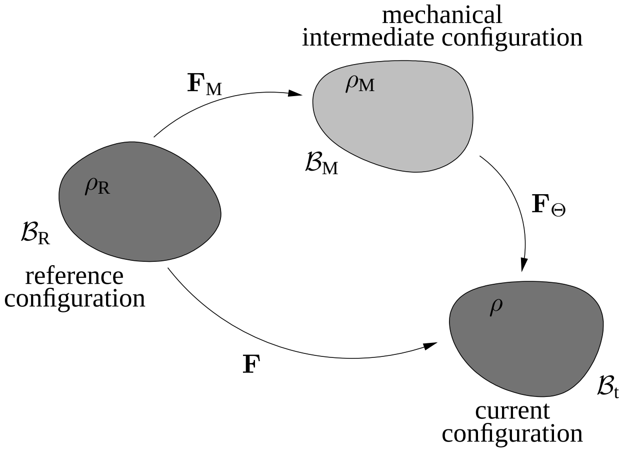


Figure 1: Sketch of multiplicative decomposition $\mathbf{F} = \mathbf{F}_{\Theta}\mathbf{F}_{\mathrm{M}}$

where $\hat{\varphi}(0) = 1$ should hold. Θ is the absolute temperature and Θ_0 defines the reference temperature. The determinant of the thermal deformation is

$$\det \mathbf{F}_{\Theta} = \hat{\varphi}(\Theta - \Theta_0) \tag{8}$$

describing the volumetric deformation caused by the temperature change $\Theta - \Theta_0$, which is chosen to be linear

$$\hat{\varphi}(\Theta - \Theta_0) := 1 + \alpha(\Theta - \Theta_0). \tag{9}$$

Of course, other proposals are possible. In [Lion, 2000b] and [Heimes, 2005] an exponential ansatz in applied as well, $\hat{\varphi}(\Theta - \Theta_0) := e^{\alpha(\Theta - \Theta_0)}$, where Eq.(9) represents the first order approximation.

$$\det \hat{\mathbf{F}}_{\mathrm{M}} = \det \mathbf{F}_{\mathrm{M}}, \qquad \det \overline{\mathbf{F}}_{\mathrm{M}} = 1 \tag{10}$$

defines the volumetric Al mechanical deformation. In view of the total deformation

$$\det \mathbf{F} = \det(\mathbf{F}_{\Theta}\mathbf{F}_{\mathrm{M}}) = (\det \mathbf{F}_{\Theta})(\det \mathbf{F}_{\mathrm{M}}) = \hat{\varphi}(\Theta - \Theta_0)(\det \mathbf{F}_{\mathrm{M}}) \tag{11}$$

holds. Accordingly, we have

$$\mathbf{F} = \hat{\mathbf{F}}\overline{\mathbf{F}} \quad \text{with} \quad \begin{cases} \hat{\mathbf{F}} & = (\varphi \ \det \mathbf{F}_{\mathrm{M}})^{1/3}\mathbf{I} \\ \overline{\mathbf{F}} & = \overline{\mathbf{F}}_{\mathrm{M}}. \end{cases} \tag{12}$$

If the densities are considered,

$$\rho_{\mathrm{R}} = (\det \mathbf{F})\rho = (\det \mathbf{F}_{\mathrm{M}})\rho_{\mathrm{M}} \tag{13}$$

defines the density in the reference configuration, ρ symbolizes the density in the current configuration, and

$$\rho_{\mathrm{M}} = (\det \mathbf{F}_{\Theta})\rho = \varphi\rho \tag{14}$$

denotes the density in the mechanical intermediate configuration.

Sometimes it is useful to introduce the abbreviation for the determinants

$$J := \det \mathbf{F} = J_{\Theta} J_{\mathrm{M}}, \tag{15}$$

$$J_{\Theta} := \det \mathbf{F}_{\Theta} = \varphi, \tag{16}$$

$$J_{\mathrm{M}} := \det \mathbf{F}_{\mathrm{M}} = J/\varphi, \tag{17}$$

which are used later for short notational purposes.

Using the imagination of a fictitious thermal unloading, similar to the case of the multiplicative decomposition of the deformation gradient into an elastic and a plastic state, see [Haupt, 1985], defines the mechanical Green strain tensor

$$\mathbf{E}_{\mathrm{M}} := \lim_{\Theta - \Theta_0 \to 0} \mathbf{E} = \frac{1}{2}(\mathbf{F}_{\mathrm{M}}^T \mathbf{F}_{\mathrm{M}} - \mathbf{I}) \tag{18}$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) \tag{19}$$

is the Green strain tensor itself. This motivates the thermal part of Greenian-type

$$\mathbf{E}_\Theta = \mathbf{E} - \mathbf{E}_\mathrm{M} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{F}_\mathrm{M}^T\mathbf{F}_\mathrm{M}) = \frac{1}{2}(\mathbf{C} - \mathbf{C}_\mathrm{M}), \tag{20}$$

or vice versa the additive decomposition

$$\mathbf{E} = \mathbf{E}_\mathrm{M} + \mathbf{E}_\Theta. \tag{21}$$

The push-forward operation $\mathbf{F}_\mathrm{M}^{-T}\mathbf{E}\mathbf{F}_\mathrm{M}^{-1}$ yields the decomposition

$$\hat{\mathbf{\Gamma}} = \hat{\mathbf{\Gamma}}_\mathrm{M} + \hat{\mathbf{\Gamma}}_\Theta \tag{22}$$

with

$$\hat{\mathbf{\Gamma}} = \mathbf{F}_\mathrm{M}^{-T}\mathbf{E}\mathbf{F}_\mathrm{M}^{-1} = \frac{1}{2}(\mathbf{F}_\Theta^T\mathbf{F}_\Theta - \mathbf{F}_\mathrm{M}^{-1}\mathbf{F}_\mathrm{M}^{-T}) = \frac{1}{2}(\varphi^{2/3}\mathbf{I} - \mathbf{B}_\mathrm{M}^{-1}) = \frac{\varphi^{2/3}}{3}(\mathbf{I} - \mathbf{B}^{-1}) \tag{23}$$

$$\hat{\mathbf{\Gamma}}_\mathrm{M} = \mathbf{F}_\mathrm{M}^{-T}\mathbf{E}_\mathrm{M}\mathbf{F}_\mathrm{M}^{-1} = \frac{1}{2}(\mathbf{I} - \mathbf{F}_\mathrm{M}^{-1}\mathbf{F}_\mathrm{M}^{-T}) = \frac{1}{2}(\mathbf{I} - \mathbf{B}_\mathrm{M}^{-1}), \tag{24}$$

$$\hat{\mathbf{\Gamma}}_\Theta = \mathbf{F}_\mathrm{M}^{-T}\mathbf{E}_\Theta\mathbf{F}_\mathrm{M}^{-1} = \frac{1}{2}(\mathbf{F}_\Theta^T\mathbf{F}_\Theta - \mathbf{I}) = \frac{1}{2}(\mathbf{C}_\Theta - \mathbf{I}) = \frac{1}{2}(\varphi^{2/3} - 1)\mathbf{I}, \tag{25}$$

where $\hat{\mathbf{\Gamma}}$, $\hat{\mathbf{\Gamma}}_\mathrm{M}$ and $\hat{\mathbf{\Gamma}}_\Theta$ measure the strains relative to the mechanical intermediate configuration. Here, the right and left Cauchy-Green tensors

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \qquad \mathbf{C}_\Theta = \mathbf{F}_\Theta^T\mathbf{F}_\Theta, \qquad \mathbf{C}_\mathrm{M} = \mathbf{F}_\mathrm{M}^T\mathbf{F}_\mathrm{M}, \tag{26}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \qquad \mathbf{B}_\Theta = \mathbf{F}_\Theta\mathbf{F}_\Theta^T, \qquad \mathbf{B}_\mathrm{M} = \mathbf{F}_\mathrm{M}\mathbf{F}_\mathrm{M}^T \tag{27}$$

are introduced. The definition of the strain measures (24) and (25) have the advantage that they are purely mechanical and purely thermal, respectively.

Additionally, strain-rate tensors on the mechanical intermediate configuration can be defined on the basis of the material time derivative of (18), (19) and (20) by the corresponding push-forward operation $\mathbf{F}_\mathrm{M}^{-T}(\ldots)\mathbf{F}_\mathrm{M}^{-1}$. This yields the strain-rate measures relative to $\mathcal{B}_\Theta$

$$\overset{\triangle}{\hat{\mathbf{\Gamma}}} = \mathbf{F}_\mathrm{M}^{-T}\dot{\mathbf{E}}\mathbf{F}_\mathrm{M}^{-1} = \dot{\hat{\mathbf{\Gamma}}} + \mathbf{L}_\mathrm{M}^T\hat{\mathbf{\Gamma}} + \hat{\mathbf{\Gamma}}\mathbf{L}_\mathrm{M}, \tag{28}$$

$$\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M} = \mathbf{F}_\mathrm{M}^{-T}\dot{\mathbf{E}}_\mathrm{M}\mathbf{F}_\mathrm{M}^{-1} = \dot{\hat{\mathbf{\Gamma}}}_\mathrm{M} + \mathbf{L}_\mathrm{M}^T\hat{\mathbf{\Gamma}}_\mathrm{M} + \hat{\mathbf{\Gamma}}_\mathrm{M}\mathbf{L}_\mathrm{M} = \frac{1}{2}(\mathbf{L}_\mathrm{M} + \mathbf{L}_\mathrm{M}^T) =: \mathbf{D}_\mathrm{M}, \tag{29}$$

$$\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\Theta = \mathbf{F}_\mathrm{M}^{-T}\dot{\mathbf{E}}_\Theta\mathbf{F}_\mathrm{M}^{-1} = \dot{\hat{\mathbf{\Gamma}}}_\Theta + \mathbf{L}_\mathrm{M}^T\hat{\mathbf{\Gamma}}_\Theta + \hat{\mathbf{\Gamma}}_\Theta\mathbf{L}_\mathrm{M}. \tag{30}$$

Obviously, the additive decomposition

$$\overset{\triangle}{\hat{\mathbf{\Gamma}}} = \overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M} + \overset{\triangle}{\hat{\mathbf{\Gamma}}}_\Theta \tag{31}$$

is inherently defined. The strain-rate $\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M}$ is purely mechanical, whereas the thermal strain-rate relative to the intermediate state can be calculated by

$$\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\Theta = \underbrace{\frac{1}{3}\hat{\varphi}'(\Theta - \Theta_0)\dot{\Theta}(t)\varphi^{-1/3}\mathbf{I}}_{\dot{\hat{\mathbf{\Gamma}}}_\Theta} + (\varphi^{2/3} - 1)\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M}, \tag{32}$$

see Eqns.(30), (25), and (29).

Next, these strain measures are used within the specific stress power to develop appropriate stress measures,

$$p = \frac{1}{\rho_\mathrm{R}}\tilde{\mathbf{T}} \cdot \dot{\mathbf{E}} = \frac{1}{\rho_\mathrm{R}}\tilde{\mathbf{T}} \cdot (\mathbf{F}_\mathrm{M}^T\overset{\triangle}{\hat{\mathbf{\Gamma}}}\mathbf{F}_\mathrm{M}) = \frac{1}{\rho_\mathrm{R}}\left(\mathbf{F}_\mathrm{M}\tilde{\mathbf{T}}\mathbf{F}_\mathrm{M}^T\right) \cdot \overset{\triangle}{\hat{\mathbf{\Gamma}}} = \frac{1}{\rho_\mathrm{R}}\hat{\mathbf{S}}_\mathrm{M} \cdot \overset{\triangle}{\hat{\mathbf{\Gamma}}} \tag{33}$$

exploiting Eq.(28) and introducing a Kirchhoff-type stress tensor relative to the mechanical intermediate configuration

$$\hat{\mathbf{S}}_\mathrm{M} = \mathbf{F}_\mathrm{M}\tilde{\mathbf{T}}\mathbf{F}_\mathrm{M}^T. \tag{34}$$

$\tilde{\mathbf{T}} = (\det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}$ defines the second Piola-Kirchhoff stress tensor and $\mathbf{T}$ the Cauchy-stresses.

Inserting the strain-rates (31) and (32) into the specific stress power (33) leads to

$$p = \frac{1}{\rho_\mathrm{R}}\hat{\mathbf{S}}_\mathrm{M}\cdot\overset{\triangle}{\hat{\mathbf{\Gamma}}} = \frac{1}{\rho_\mathrm{R}}\hat{\mathbf{S}}_\mathrm{M}\cdot(\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M} + \overset{\triangle}{\hat{\mathbf{\Gamma}}}_\Theta) = \frac{1}{\rho_\mathrm{R}}\varphi^{2/3}\hat{\mathbf{S}}_\mathrm{M}\cdot\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M} + \frac{\hat{\varphi}'(\Theta-\Theta_0)\varphi^{-1/3}\dot{\Theta}(t)}{3\rho_\mathrm{R}}(\mathrm{tr}\,\hat{\mathbf{S}}_\mathrm{M}). \tag{35}$$

In view of thermo-mechanical processes the Clausius-Duhem inequality has to be fulfilled requiring the stress power

$$-\dot{\psi} - \dot{\Theta}s + \frac{1}{\rho_\mathrm{R}}\tilde{\mathbf{T}}\cdot\dot{\mathbf{E}} - \frac{\vec{q}}{\rho\Theta}\cdot\mathrm{grad}\,\Theta = -\dot{\psi} - \dot{\Theta}s + \frac{1}{\rho_\mathrm{R}}\mathbf{S}_\mathrm{M}\cdot\overset{\triangle}{\hat{\mathbf{\Gamma}}} - \frac{\vec{q}}{\rho\Theta}\cdot\mathrm{grad}\,\Theta \geq 0, \tag{36}$$

where ψ defines the specific free energy, s the entropy, and $\vec{q}$ the heat flux vector. In the following, the proposal of [Lion, 2000b] and [Heimes, 2005] is taken up where the free-energy depends on the mechanical deformation $\mathbf{E}_\mathrm{M}$ and the temperature Θ,

$$\psi(\mathbf{E}_\mathrm{M},\Theta) = \psi_\mathrm{M}(\mathbf{C}_\mathrm{M},\Theta) + \psi_\Theta(\Theta). \tag{37}$$

Of course, the first natural assumption would be $\psi(\mathbf{C}_\mathrm{M},\Theta) = \psi_\mathrm{M}(\mathbf{C}_\mathrm{M}) + \psi_\Theta(\Theta)$, i.e. there are clear assignments of mechanical and thermal induced stresses. However, in rubber elasticity it turns out experimentally that the stress state depends linearly on the temperature. Thus, $\psi_\mathrm{M}(\mathbf{C}_\mathrm{M},\Theta)$ is assumed to be a function of the temperature as well. The dependence of the mechanical part is assumed to be linear in the temperature

$$\psi_\mathrm{M}(\mathbf{C}_\mathrm{M},\Theta) = \frac{\Theta}{\Theta_0}\overline{\psi}_\mathrm{M}(\mathbf{C}_\mathrm{M}) \tag{38}$$

with

$$\overline{\psi}_\mathrm{M}(\mathbf{C}_\mathrm{M}) = U(J_\mathrm{M}) + \overline{v}(\overline{\mathbf{C}}_\mathrm{M}) = U(J_\mathrm{M}) + w(\mathrm{I}_{\overline{\mathbf{C}}_\mathrm{M}},\mathrm{II}_{\overline{\mathbf{C}}_\mathrm{M}}), \tag{39}$$

$\overline{v}(\overline{\mathbf{C}}_\mathrm{M}) = w(\mathrm{I}_{\overline{\mathbf{C}}_\mathrm{M}},\mathrm{II}_{\overline{\mathbf{C}}_\mathrm{M}})$. Here, the mechanical deformation is decomposed into volume-changing and preserving parts, defined by

$$J_\mathrm{M} = \det\mathbf{F}_\mathrm{M} = (\det\mathbf{C}_\mathrm{M})^{1/2} \tag{40}$$

of Eq.(17) using the unimodular, mechanical right Cauchy-Green tensor

$$\overline{\mathbf{C}}_\mathrm{M} = (\det\mathbf{C}_\mathrm{M})^{-1/3}\mathbf{C}_\mathrm{M}, \qquad \det\overline{\mathbf{C}}_\mathrm{M} = 1. \tag{41}$$

The thermal part of the strain-energy (37) is defined by

$$\psi_\Theta(\Theta) = \frac{c_p}{\rho_\mathrm{R}}\left(\left((\Theta-\Theta_0) - \Theta\ln\frac{\Theta}{\Theta_0}\right)(1-k_p\Theta_0) - \frac{1}{2}k_p(\Theta^2-\Theta_0^2)\right), \tag{42}$$

see [Heimes, 2005].

The evaluation of the material time-derivative of the free energy ψ, see Eq.(37),

$$\dot{\psi}(\mathbf{E}_\mathrm{M},\Theta) = \left(\frac{1}{\Theta_0}\overline{\psi}_\mathrm{M}(\mathbf{C}_\mathrm{M})\right)\dot{\Theta} + \frac{\Theta}{\Theta_0}\frac{\mathrm{d}\overline{\psi}_\mathrm{M}(\mathbf{C}_\mathrm{M})}{\mathrm{d}\mathbf{C}_\mathrm{M}}\cdot\dot{\mathbf{C}}_\mathrm{M} + \psi_\Theta'(\Theta)\dot{\Theta}, \tag{43}$$

is required in the Clausius-Duhem inequality (36) yielding, by means of definition (34) and the time derivative of (18) expressed by the mechanical right Cauchy-Green tensor $(26)_3$, $\dot{\mathbf{C}}_\mathrm{M} = 2\mathbf{F}_\mathrm{M}^T\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M}\mathbf{F}_\mathrm{M}$,

$$\begin{aligned}&\left(\frac{1}{\rho_\mathrm{R}}\varphi^{2/3}\hat{\mathbf{S}}_\mathrm{M} - 2\frac{\Theta}{\Theta_0}\mathbf{F}_\mathrm{M}\frac{\mathrm{d}\overline{\psi}_\mathrm{M}(\mathbf{C}_\mathrm{M})}{\mathrm{d}\mathbf{C}_\mathrm{M}}\mathbf{F}_\mathrm{M}^T\right)\cdot\overset{\triangle}{\hat{\mathbf{\Gamma}}}_\mathrm{M}\\ &\qquad + \left(-s - \left(\frac{1}{\Theta_0}\overline{\psi}_\mathrm{M}(\mathbf{C}_\mathrm{M}) + \psi_\Theta'(\Theta)\right) + \frac{1}{3\rho_\mathrm{R}}\varphi^{-1/3}\hat{\varphi}'(\Theta-\Theta_0)(\mathrm{tr}\,\hat{\mathbf{S}}_\mathrm{M})\right)\dot{\Theta}\\ &\qquad - \frac{\vec{q}}{\rho\Theta}\cdot\mathrm{grad}\,\Theta \geq 0\end{aligned} \tag{44}$$

Exploiting arbitrary mechanical strain and temperature paths commonly implies the three following expressions:

$$\hat{\mathbf{S}}_{\mathrm{M}} = \frac{2\rho_{\mathrm{R}}}{\varphi^{2/3}}\frac{\Theta}{\Theta_0}\mathbf{F}_{\mathrm{M}}\frac{\mathrm{d}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}})}{\mathrm{d}\mathbf{C}_{\mathrm{M}}}\mathbf{F}_{\mathrm{M}}^{T}, \tag{45}$$

$$s = -\frac{1}{\Theta_0}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}}) - \psi'_{\Theta}(\Theta) + \frac{1}{3\rho_{\mathrm{R}}}\varphi^{-1/3}\hat{\varphi}'(\Theta-\Theta_0)(\operatorname{tr}\hat{\mathbf{S}}_{\mathrm{M}}), \tag{46}$$

$$\vec{q} = -\kappa\,\mathrm{grad}\,\Theta, \qquad \kappa \geq 0 \tag{47}$$

In the following, the stress tensor and the entropy are expressed by quantities relative to the current configuration. The particular strain-energy function (39) yields for the derivatives in the elasticity relation (45) the terms

$$\frac{\mathrm{d}U((\det\mathbf{C}_{\mathrm{M}})^{1/2})}{\mathrm{d}\mathbf{C}_{\mathrm{M}}} = \frac{1}{2}J_{\mathrm{M}}U'(J_{\mathrm{M}})\mathbf{C}_{\mathrm{M}}^{-1} \tag{48}$$

and

$$\frac{\mathrm{d}\overline{v}(\overline{\mathbf{C}}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}}))}{\mathrm{d}\mathbf{C}_{\mathrm{M}}} = \left[\frac{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}}{\mathrm{d}\mathbf{C}_{\mathrm{M}}}\right]^{T}\frac{\mathrm{d}\overline{v}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}} \tag{49}$$

with

$$\left[\frac{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}}{\mathrm{d}\mathbf{C}_{\mathrm{M}}}\right]^{T} = (\det\mathbf{C}_{\mathrm{M}})^{-1/3}\left[\mathcal{I} - \frac{1}{3}(\mathbf{C}_{\mathrm{M}}^{-1}\otimes\mathbf{C}_{\mathrm{M}})\right] = J_{\mathrm{M}}^{-2/3}\left[\mathcal{I} - \frac{1}{3}(\overline{\mathbf{C}}_{\mathrm{M}}^{-1}\otimes\overline{\mathbf{C}}_{\mathrm{M}})\right]. \tag{50}$$

$\mathcal{I}$ defines the identity tensor of fourth order,

$$\mathcal{I} = [\mathbf{I}\otimes\mathbf{I}]^{T_{23}} = \delta_{ik}\delta_{jl}\vec{e}_i\otimes\vec{e}_j\otimes\vec{e}_k\otimes\vec{e}_l, \tag{51}$$

here defined in Cartesian coordinates, $\mathcal{I}\mathbf{A} = \mathbf{A}$. Obviously, $\mathbf{C}_{\mathrm{M}}^{-1}\otimes\mathbf{C}_{\mathrm{M}} = \overline{\mathbf{C}}_{\mathrm{M}}^{-1}\otimes\overline{\mathbf{C}}_{\mathrm{M}}$ holds. Caused by the particular dependence on the invariants of the mechanical unimodular right Cauchy-Green tensors, the application of the chain-rule leads to

$$\frac{\mathrm{d}\overline{v}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}} = \frac{\partial w}{\partial \mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}}}\frac{\mathrm{d}\mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}} + \frac{\partial w}{\partial \mathrm{II}_{\overline{\mathbf{C}}_{\mathrm{M}}}}\frac{\mathrm{d}\mathrm{II}_{\overline{\mathbf{C}}_{\mathrm{M}}}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}} = (w_1 + w_2\mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}})\mathbf{I} - w_2\overline{\mathbf{C}}_{\mathrm{M}} \tag{52}$$

with

$$w_1(\mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}},\mathrm{II}_{\overline{\mathbf{C}}_{\mathrm{M}}}) = \frac{\partial w}{\partial \mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}}} \qquad \text{and} \qquad w_2(\mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}},\mathrm{II}_{\overline{\mathbf{C}}_{\mathrm{M}}}) = \frac{\partial w}{\partial \mathrm{II}_{\overline{\mathbf{C}}_{\mathrm{M}}}}. \tag{53}$$

In other words, we have

$$\frac{\mathrm{d}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}})}{\mathrm{d}\mathbf{C}_{\mathrm{M}}} = J_{\mathrm{M}}U'(J_{\mathrm{M}})\mathbf{C}_{\mathrm{M}}^{-1} + 2J_{\mathrm{M}}^{-2/3}\left[\mathcal{I} - \frac{1}{3}\overline{\mathbf{C}}_{\mathrm{M}}^{-1}\otimes\overline{\mathbf{C}}_{\mathrm{M}}\right]\frac{\mathrm{d}\overline{v}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}} = \tag{54}$$

$$= J_{\mathrm{M}}^{1/3}U'(J_{\mathrm{M}})\overline{\mathbf{C}}_{\mathrm{M}}^{-1} + 2J_{\mathrm{M}}^{-2/3}\left((w_1 + w_2\mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}})\mathbf{I} - w_2\overline{\mathbf{C}}_{\mathrm{M}} - \frac{1}{3}(w_1\mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}} + 2w_2\mathrm{II}_{\overline{\mathbf{C}}_{\mathrm{M}}})\overline{\mathbf{C}}_{\mathrm{M}}^{-1}\right). \tag{55}$$

In the following, these expressions are used to express the elasticity relation relative to the current and the reference configuration. First, the push-forward operation of the second Piola-Kirchhoff tensor yields

$$\begin{aligned}\mathbf{S} &= \mathbf{F}\tilde{\mathbf{T}}\mathbf{F}^{T} = \mathbf{F}(\mathbf{F}_{\mathrm{M}}^{-1}\hat{\mathbf{S}}_{\mathrm{M}}\mathbf{F}_{\mathrm{M}}^{-T})\mathbf{F}^{T} = \\ &= \mathbf{F}_{\Theta}\mathbf{F}_{\mathrm{M}}(\mathbf{F}_{\mathrm{M}}^{-1}\hat{\mathbf{S}}_{\mathrm{M}}\mathbf{F}_{\mathrm{M}}^{-T})\mathbf{F}_{\mathrm{M}}^{-T}\mathbf{F}_{\Theta}^{T} = \\ &= \mathbf{F}_{\Theta}\hat{\mathbf{S}}_{\mathrm{M}}\mathbf{F}_{\Theta}^{T} = \varphi^{2/3}\hat{\mathbf{S}}_{\mathrm{M}}\end{aligned} \tag{56}$$

where use is made of the decomposition (2) and Eq.(34). $\mathbf{S} = (\det\mathbf{F})\mathbf{T}$ defines the weighted Cauchy tensor, i.e. the Kirchhoff stresses. Comparing (56) with (45) yields

$$\mathbf{S} = 2\rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}\mathbf{F}_{\mathrm{M}}\frac{\mathrm{d}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}})}{\mathrm{d}\mathbf{C}_{\mathrm{M}}}\mathbf{F}_{\mathrm{M}}^{T} = \tag{57}$$

$$\begin{aligned}&= \rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}J_{\mathrm{M}}U'(J_{\mathrm{M}})\mathbf{I} + 2\rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}J_{\mathrm{M}}^{-2/3}\left[\mathbf{F}_{\mathrm{M}}\otimes\mathbf{F}_{\mathrm{M}}\right]^{T_{23}}\left[\mathcal{I} - \frac{1}{3}\mathbf{C}_{\mathrm{M}}^{-1}\otimes\mathbf{C}_{\mathrm{M}}\right]\frac{\mathrm{d}\overline{v}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}} = \\ &= \rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}J_{\mathrm{M}}U'(J_{\mathrm{M}})\mathbf{I} + 2\rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}\left[\mathcal{I} - \frac{1}{3}\mathbf{I}\otimes\mathbf{I}\right]\left[\overline{\mathbf{F}}_{\mathrm{M}}\otimes\overline{\mathbf{F}}_{\mathrm{M}}\right]^{T_{23}}\frac{\mathrm{d}\overline{v}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}} = \\ &= \rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}J_{\mathrm{M}}U'(J_{\mathrm{M}})\mathbf{I} + 2\rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}\left(\overline{\mathbf{F}}_{\mathrm{M}}\frac{\mathrm{d}\overline{v}}{\mathrm{d}\overline{\mathbf{C}}_{\mathrm{M}}}\overline{\mathbf{F}}_{\mathrm{M}}^{T}\right)^{D}.\end{aligned} \tag{58}$$

Exploiting the property

$$[\mathbf{F}_\mathrm{M} \otimes \mathbf{F}_\mathrm{M}]^{T_{23}} \left[\mathcal{I} - \frac{1}{3}\mathbf{C}_\mathrm{M}^{-1} \otimes \mathbf{C}_\mathrm{M}\right] = \left[\mathcal{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I}\right] [\mathbf{F}_\mathrm{M} \otimes \mathbf{F}_\mathrm{M}]^{T_{23}}\,, \tag{59}$$

where

$$\mathcal{D} = \mathcal{I} - \frac{1}{3}\mathbf{I} \otimes \mathbf{I} \tag{60}$$

defines the deviator operator $\mathcal{D}\mathbf{A} = \mathbf{A}^D = \mathbf{A} - \frac{1}{3}(\mathrm{tr}\,\mathbf{A})\mathbf{I}$, yields under the assumption of isotropy the Kirchhoff stress tensor

$$\mathbf{S} = \rho_\mathrm{R}\frac{\Theta}{\Theta_0}J_\mathrm{M}U'(J_\mathrm{M})\mathbf{I} + 2\rho_\mathrm{R}\frac{\Theta}{\Theta_0}\left(\overline{\mathbf{B}}_\mathrm{M}\frac{\mathrm{d}\overline{\upsilon}}{\mathrm{d}\overline{\mathbf{B}}_\mathrm{M}}\right)^D\,. \tag{61}$$

Finally, the Cauchy stress tensor reads

$$\mathbf{T} = \frac{\rho_\mathrm{R}}{\varphi}\frac{\Theta}{\Theta_0}U'(J/\varphi)\mathbf{I} + \frac{2\rho_\mathrm{R}}{J}\frac{\Theta}{\Theta_0}\left(\frac{\mathrm{d}\overline{\upsilon}}{\mathrm{d}\overline{\mathbf{B}}_\mathrm{M}}\overline{\mathbf{B}}_\mathrm{M}\right)^D\,. \tag{62}$$

If one considers the mechanical unimodular left Cauchy-Green tensor

$$\overline{\mathbf{B}}_\mathrm{M} = J_\mathrm{M}^{-2/3}\mathbf{B}_\mathrm{M} = (J/\varphi)^{-2/3}\varphi^{-2/3}\mathbf{B} = J^{-2/3}\mathbf{B} = \overline{\mathbf{B}}, \tag{63}$$

Eq.(62) can be expressed by

$$\mathbf{T} = \frac{\rho_\mathrm{R}}{\varphi}\frac{\Theta}{\Theta_0}U'(J/\varphi)\mathbf{I} + \frac{2\rho_\mathrm{R}}{J}\frac{\Theta}{\Theta_0}\left(\frac{\mathrm{d}\overline{\upsilon}}{\mathrm{d}\overline{\mathbf{B}}}\overline{\mathbf{B}}\right)^D\,, \tag{64}$$

i.e. only the volumetric part of the strain-energy function $U(J_\mathrm{M}) = U(J/\varphi)$ depends on the temperature by the factor $\hat{\varphi}^{-1}(\Theta - \Theta_0)U'(J/\hat{\varphi}(\Theta - \Theta_0))$, and the overall stress state is assumed to increase linearly with Θ/Θ_0. Since $\varphi \approx 1$ holds, the curves are only scarcely influenced.

Furthermore, the entropy (46) reads

$$\begin{aligned} s &= -\frac{1}{\Theta_0}\overline{\psi}_\mathrm{M}(\mathbf{C}_\mathrm{M}) - \psi'_\Theta(\Theta) + \frac{1}{3\rho_\mathrm{R}}\varphi^{-1/3}\hat{\varphi}'(\Theta - \Theta_0)(\mathrm{tr}\,\mathbf{S}) = \\ &= -\frac{1}{\Theta_0}\left(U\left(\frac{J}{\varphi}\right) + \hat{\upsilon}(\overline{\mathbf{C}})\right) - \psi'_\Theta(\Theta) + \frac{\Theta}{\Theta_0}\frac{\hat{\varphi}'(\Theta - \Theta_0)}{\varphi^2}JU'\left(\frac{J}{\varphi}\right) \end{aligned} \tag{65}$$

using relation (56).

3 Strain and stress measures of decomposition $\mathbf{F} = \mathbf{F}_\mathbf{M}\mathbf{F}_\Theta$

In this section the decomposition (1) is considered, see Fig. 2. However, we extend again the investigations to the split into a volume-preserving and a volume-changing part

$$\mathbf{F} = \mathbf{F}_\mathrm{M}\mathbf{F}_\Theta = \hat{\mathbf{F}}_\mathrm{M}\overline{\mathbf{F}}_\mathrm{M}\mathbf{F}_\Theta, \tag{66}$$

defined in Eqns.(4), (5) and (6). The Green strain tensor measuring the thermal deformation is defined by a fictive mechanical unloading

$$\tilde{\mathbf{E}}_\Theta := \lim_{\|\mathbf{F}_\mathrm{M}\| \to 0} \mathbf{E} = \frac{1}{2}(\mathbf{F}_\Theta^T\mathbf{F}_\Theta - \mathbf{I}) \tag{67}$$

leading to

$$\tilde{\mathbf{E}}_\Theta = \frac{1}{2}(\varphi^{2/3} - 1)\mathbf{I}, \tag{68}$$

where the thermal part of the deformation gradient (7) is inserted. Accordingly, the mechanical part reads

$$\tilde{\mathbf{E}}_\mathrm{M} = \mathbf{E} - \tilde{\mathbf{E}}_\Theta = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{F}_\Theta^T\mathbf{F}_\Theta) = \frac{1}{2}(\mathbf{C} - \varphi^{2/3}\mathbf{I}). \tag{69}$$

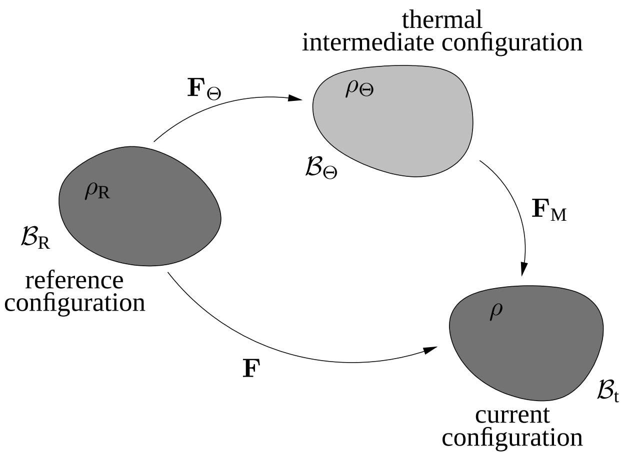


Figure 2: Sketch of multiplicative decomposition (1)

In order to obtain the strain measures on the thermal intermediate configuration, the push-forward operation $\hat{\gamma} = \mathbf{F}_\Theta^{-T}\mathbf{E}\mathbf{F}_\Theta^{-1}$ is introduced yielding

$$\hat{\gamma} = \frac{1}{2}(\mathbf{F}_\mathrm{M}^T\mathbf{F}_\mathrm{M} - \mathbf{F}_\Theta^{-T}\mathbf{F}_\Theta^{-1}) = \frac{1}{2}(\mathbf{C}_\mathrm{M} - \mathbf{B}_\Theta^{-1}) = \frac{1}{2}(\mathbf{C}_\mathrm{M} - \varphi^{-2/3}\mathbf{I}) \tag{70}$$

$$\hat{\gamma}_\mathrm{M} = \mathbf{F}_\Theta^{-T}\tilde{\mathbf{E}}_\mathrm{M}\mathbf{F}_\Theta^{-1} = \frac{1}{2}(\mathbf{F}_\mathrm{M}^T\mathbf{F}_\mathrm{M} - \mathbf{I}) = \frac{1}{2}(\mathbf{C}_\mathrm{M} - \mathbf{I}) \tag{71}$$

$$\hat{\gamma}_\Theta = \mathbf{F}_\Theta^{-T}\tilde{\mathbf{E}}_\Theta\mathbf{F}_\Theta^{-1} = \frac{1}{2}(\mathbf{I} - \mathbf{F}_\Theta^{-T}\mathbf{F}_\Theta^{-1}) = \frac{1}{2}(1 - \varphi^{-2/3})\mathbf{I} \tag{72}$$

i.e.

$$\hat{\gamma} = \hat{\gamma}_\mathrm{M} + \hat{\gamma}_\Theta. \tag{73}$$

Using

$$\dot{\mathbf{F}}_\Theta = \frac{1}{3}\dot{\varphi}\varphi^{-2/3}\mathbf{I} \tag{74}$$

and

$$\mathbf{L}_\Theta = \dot{\mathbf{F}}_\Theta\mathbf{F}_\Theta^{-1} = \frac{\dot{\varphi}}{3\varphi}\mathbf{I} \tag{75}$$

the strain-rate tensors relative to the reference configuration read

$$\dot{\mathbf{E}} = \frac{1}{2}\dot{\mathbf{C}} = \dot{\tilde{\mathbf{E}}}_\Theta + \dot{\tilde{\mathbf{E}}}_\mathrm{M} \tag{76}$$

$$\dot{\tilde{\mathbf{E}}}_\mathrm{M} = \frac{1}{2}(\dot{\mathbf{C}} - \frac{2}{3}\dot{\varphi}\varphi^{-1/3}\mathbf{I}) = \frac{1}{2}\dot{\mathbf{C}} - \frac{1}{3}\dot{\varphi}\varphi^{-1/3}\mathbf{I} \tag{77}$$

$$\dot{\tilde{\mathbf{E}}}_\Theta = \frac{1}{2}(\dot{\mathbf{F}}_\Theta^T\mathbf{F}_\Theta + \mathbf{F}_\Theta^T\dot{\mathbf{F}}_\Theta) = \frac{1}{3}\dot{\varphi}\varphi^{-1/3}\mathbf{I} \tag{78}$$

and the push-forward operation to the thermal intermediate configuration yields

$$\overset{\blacktriangle}{\hat{\gamma}} = \mathbf{F}_\Theta^{-T}\dot{\mathbf{E}}\mathbf{F}_\Theta^{-1} \qquad = \dot{\hat{\gamma}} + \mathbf{L}_\Theta^T\hat{\gamma} + \hat{\gamma}\mathbf{L}_\Theta = \dot{\hat{\gamma}} + \frac{2}{3}\frac{\dot{\varphi}}{\varphi}\hat{\gamma} \tag{79}$$

$$\overset{\blacktriangle}{\hat{\gamma}}_\mathrm{M} = \mathbf{F}_\Theta^{-T}\dot{\tilde{\mathbf{E}}}_\mathrm{M}\mathbf{F}_\Theta^{-1} = \dot{\hat{\gamma}}_\mathrm{M} + \mathbf{L}_\Theta^T\hat{\gamma}_\mathrm{M} + \hat{\gamma}_\mathrm{M}\mathbf{L}_\Theta = \dot{\hat{\gamma}}_\mathrm{M} + \frac{2}{3}\frac{\dot{\varphi}}{\varphi}\hat{\gamma}_\mathrm{M} = \frac{1}{2}\dot{\mathbf{C}}_\mathrm{M} + \frac{1}{3}\frac{\dot{\varphi}}{\varphi}\mathbf{C}_\mathrm{M} - \frac{1}{3}\frac{\dot{\varphi}}{\varphi}\mathbf{I} \tag{80}$$

$$\overset{\blacktriangle}{\hat{\gamma}}_\Theta = \mathbf{F}_\Theta^{-T}\dot{\tilde{\mathbf{E}}}_\Theta\mathbf{F}_\Theta^{-1} = \dot{\hat{\gamma}}_\Theta + \mathbf{L}_\Theta^T\hat{\gamma}_\Theta + \hat{\gamma}_\Theta\mathbf{L}_\Theta = \mathbf{D}_\Theta = \mathbf{L}_\Theta = \frac{\dot{\varphi}}{3\varphi}\mathbf{I} \tag{81}$$

with the Oldroyd strain-rate

$$\overset{\blacktriangle}{\hat{\gamma}} = \overset{\blacktriangle}{\hat{\gamma}}_\mathrm{M} + \overset{\blacktriangle}{\hat{\gamma}}_\Theta. \tag{82}$$

In the following, the specific stress power has to be evaluated leading to

$$p = \frac{1}{\rho_{\mathrm{R}}}\tilde{\mathbf{T}} \cdot \dot{\mathbf{E}} = \frac{1}{\rho_{\mathrm{R}}}\tilde{\mathbf{T}} \cdot (\mathbf{F}_\Theta^T \overset{\blacktriangle}{\hat{\gamma}} \mathbf{F}_\Theta) = \frac{1}{\rho_{\mathrm{R}}}(\mathbf{F}_\Theta \tilde{\mathbf{T}} \mathbf{F}_\Theta^T) \cdot \overset{\blacktriangle}{\hat{\gamma}} = \frac{1}{\rho_{\mathrm{R}}}\hat{\mathbf{S}}_\Theta \cdot \overset{\blacktriangle}{\hat{\gamma}}, \tag{83}$$

with the stress tensor $\hat{\mathbf{S}}_\Theta$ relative to the thermal intermediate configuration

$$\hat{\mathbf{S}}_\Theta = \mathbf{F}_\Theta \tilde{\mathbf{T}} \mathbf{F}_\Theta^T = \varphi^{2/3}\tilde{\mathbf{T}}. \tag{84}$$

If one inserts the additive decomposition (82) into (83) and exploits Eqns.(80) and (81), the following expression for the stress power results,

$$p = \frac{1}{\rho_{\mathrm{R}}}\hat{\mathbf{S}}_\Theta \cdot (\overset{\blacktriangle}{\hat{\gamma}}_{\mathrm{M}} + \overset{\blacktriangle}{\hat{\gamma}}_\Theta) = \frac{1}{\rho_{\mathrm{R}}}\hat{\mathbf{S}}_\Theta \cdot \left(\frac{1}{2}\dot{\mathbf{C}}_{\mathrm{M}} + \frac{1}{3}\frac{\dot{\varphi}}{\varphi}\mathbf{C}_{\mathrm{M}}\right). \tag{85}$$

Inserting this expression with the material time derivative of the specific strain-energy function (43) into the Clausius-Duhem inequality (36) yields

$$\left(-s - \frac{1}{\Theta_0}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}}) - \psi_\Theta'(\Theta) + \frac{\hat{\varphi}'(\Theta - \Theta_0)}{3\rho_{\mathrm{R}}\varphi}(\hat{\mathbf{S}}_\Theta \cdot \mathbf{C}_{\mathrm{M}})\right)\dot{\Theta} + \left(\frac{1}{2\rho_{\mathrm{R}}}\hat{\mathbf{S}}_\Theta - \frac{\Theta}{\Theta_0}\frac{\mathrm{d}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}})}{\mathrm{d}\mathbf{C}_{\mathrm{M}}}\right) \cdot \dot{\mathbf{C}}_{\mathrm{M}} - \frac{\vec{q}}{\rho\Theta} \cdot \operatorname{grad}\Theta \geq 0. \tag{86}$$

For independent thermal and mechanical processes a sufficient condition to satisfy the Clausius-Duhem inequality is

$$\hat{\mathbf{S}}_\Theta = 2\rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}\frac{\mathrm{d}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}})}{\mathrm{d}\mathbf{C}_{\mathrm{M}}} \tag{87}$$

$$s = -\frac{1}{\Theta_0}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}}) - \psi_\Theta'(\Theta) + \frac{\hat{\varphi}'(\Theta - \Theta_0)}{3\rho_{\mathrm{R}}\varphi}(\hat{\mathbf{S}}_\Theta \cdot \mathbf{C}_{\mathrm{M}}) \tag{88}$$

$$\vec{q} = -\lambda \operatorname{grad}\Theta. \tag{89}$$

The push-forward operation of the stress tensor (87), see also Eq.(84), yields

$$\mathbf{S} = \mathbf{F}\tilde{\mathbf{T}}\mathbf{F}^T = \mathbf{F}_{\mathrm{M}}\hat{\mathbf{S}}_\Theta\mathbf{F}_{\mathrm{M}}^T = 2\rho_{\mathrm{R}}\frac{\Theta}{\Theta_0}\mathbf{F}_{\mathrm{M}}\frac{\mathrm{d}\overline{\psi}_{\mathrm{M}}(\mathbf{C}_{\mathrm{M}})}{\mathrm{d}\mathbf{C}_{\mathrm{M}}}\mathbf{F}_{\mathrm{M}}^T, \tag{90}$$

i.e. exactly the same expression as developed in Eq.(57). In other words, for an isotropic and volumetric thermal expansion according to Eq.(7), the decomposition (1) and (2) are equivalent. This holds also for the entropy (88), because of $\hat{\mathbf{S}}_\Theta \cdot \mathbf{C}_{\mathrm{M}} = \mathbf{F}_{\mathrm{M}}\hat{\mathbf{S}}_\Theta\mathbf{F}_{\mathrm{M}}^T \cdot \mathbf{I} = \operatorname{tr}\mathbf{S}$.

4 Classical approach

The multiplicative decomposition of the deformation gradient into a mechanical and a thermal part can also be seen as a motivation of the structure of the free energy. In order to show this, we proceed as follows. Similarly to Eq.(63) it follows $\overline{\mathbf{C}}_{\mathrm{M}} = \overline{\mathbf{C}}$. For the strain-energy function (37) with the specific forms (38) and (39) one obtains

$$\hat{\psi}(J, \overline{\mathbf{C}}, \Theta) := \frac{\Theta}{\Theta_0}\left(U(J_{\mathrm{M}}) + w(\mathrm{I}_{\overline{\mathbf{C}}_{\mathrm{M}}}, \mathrm{II}_{\overline{\mathbf{C}}_{\mathrm{M}}})\right) + \psi_\Theta(\Theta) = \tag{91}$$

$$= \frac{\Theta}{\Theta_0}U(J/\varphi) + \frac{\Theta}{\Theta_0}w(\mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}}) + \psi_\Theta(\Theta). \tag{92}$$

The second Piola-Kirchhoff stress tensor reads in the case of the classical thermo-viscoelasticity (which is not based on the decomposition of the deformation gradient)

$$\frac{1}{\rho_{\mathrm{R}}}\tilde{\mathbf{T}} = \frac{\partial\hat{\psi}(J, \overline{\mathbf{C}}, \Theta)}{\partial\mathbf{C}}, \tag{93}$$

see [Haupt, 2000, Sec. 13.2]. If one calculates this derivative, one arrives at

$$\frac{1}{\rho_{\mathrm{R}}}\tilde{\mathbf{T}} = \frac{\Theta}{\Theta_0} J U'(J/\varphi) + 2\frac{\Theta}{\Theta_0}\frac{\mathrm{d}w}{\mathrm{d}\mathbf{C}}. \tag{94}$$

In analogy to Eqns.(48)-(55) the second Piola-Kirchhoff tensor

$$\frac{1}{\rho_{\mathrm{R}}}\tilde{\mathbf{T}} = \frac{\Theta}{\Theta_0}\left(JU'(J/\varphi)\mathbf{C}^{-1} + 2J^{-2/3}\left((w_1 + w_2 \mathrm{I}_{\overline{\mathbf{C}}})\mathbf{I} - w_2\overline{\mathbf{C}} - \frac{1}{3}(w_1 \mathrm{I}_{\overline{\mathbf{C}}} + 2w_2 \mathrm{II}_{\overline{\mathbf{C}}})\overline{\mathbf{C}}^{-1}\right)\right) \tag{95}$$

follows, exactly leading to the same Cauchy stresses as before. The same holds for the entropy

$$s = -\frac{\partial\hat{\psi}(J,\overline{\mathbf{C}},\Theta)}{\partial\Theta} = -\frac{1}{\Theta_0}(U(J/\varphi) + w(\mathrm{I}_{\overline{\mathbf{C}}}, \mathrm{II}_{\overline{\mathbf{C}}})) + J\hat{\varphi}'(\Theta-\Theta_0)J^{-2}\frac{\Theta}{\Theta_0}U'(J/\varphi) - \frac{\partial\psi_\Theta}{\partial\Theta}, \tag{96}$$

see Eq.(65). Thus, the decompositions can be seen as motivations of the free energy form.

5 Simple investigations

In the following, two simple analytical examples are investigated. First, the uniaxial tensile test for constant temperatures is looked for. Second, the uniaxial tensile/compression test is essentially influenced by the "volumetric" strain-energy function $U(J_{\mathrm{M}})$, see [Hartmann, 2003, Hartmann and Neff, 2003, Ehlers and Eipper, 1998]. Thus, this behavior is studied as well.

5.1 Uniaxial tensile-compression test with constant temperatures

In the first investigation the uniaxial tensile and compression test is assumed with constant temperature. In this case the deformation gradient has the representation

$$\mathbf{F} = \begin{bmatrix} \lambda & & \\ & \lambda_Q & \\ & & \lambda_Q \end{bmatrix} \vec{e}_i \otimes \vec{e}_j, \qquad i,j = 1,2,3, \tag{97}$$

where λ defines the prescribed axial stretch and λ_Q the unknown lateral stretch in a bar. The unimodular left Cauchy-Green tensor of Eq.(63) reads

$$\overline{\mathbf{B}} = (\lambda\lambda_Q^2)^{-2/3}\begin{bmatrix} \lambda^2 & & \\ & \lambda_Q^2 & \\ & & \lambda_Q^2 \end{bmatrix} \vec{e}_i \otimes \vec{e}_j, \qquad i,j = 1,2,3, \tag{98}$$

with the determinant of the deformation gradient

$$J = \det\mathbf{F} = \lambda\lambda_Q^2. \tag{99}$$

The stress state is given by

$$\mathbf{T} = \begin{bmatrix} \sigma & & \\ & 0 & \\ & & 0 \end{bmatrix} \vec{e}_i \otimes \vec{e}_j, \qquad i,j = 1,2,3, \tag{100}$$

which has to be inserted into the thermo-elasticity relation (64). This leads with

$$\begin{aligned}\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{\mathbf{B}}}\overline{\mathbf{B}} &= (w_1 + w_2\mathrm{I}_{\overline{\mathbf{B}}})\overline{\mathbf{B}} - w_2\overline{\mathbf{B}}^2 = \\ &= \begin{bmatrix} (w_1 + w_2\mathrm{I}_{\overline{\mathbf{B}}})\overline{B} - w_2\overline{B}^2 & & \\ & (w_1 + w_2\mathrm{I}_{\overline{\mathbf{B}}})\overline{B}_Q - w_2\overline{B}_Q^2 & \\ & & (w_1 + w_2\mathrm{I}_{\overline{\mathbf{B}}})\overline{B}_Q - w_2\overline{B}_Q^2 \end{bmatrix} \vec{e}_i \otimes \vec{e}_j,\end{aligned}$$

to the deviator

$$\left(\frac{\mathrm{d}\overline{v}}{\mathrm{d}\overline{\mathbf{B}}}\overline{\mathbf{B}}\right)^D = \frac{2}{3}\left((w_1 + w_2\mathrm{I}_{\overline{\mathbf{B}}})(\overline{B} - \overline{B}_Q) - w_2(\overline{B}^2 - \overline{B}_Q^2)\right)\begin{bmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & -\frac{1}{2}\end{bmatrix}\vec{e}_i \otimes \vec{e}_j. \tag{101}$$

Here, the abbreviations

$$\overline{B} := (\lambda\lambda_Q^2)^{-2/3}\lambda^2 \qquad \text{and} \qquad \overline{B}_Q := (\lambda\lambda_Q^2)^{-2/3}\lambda_Q^2$$

are introduced, see Eq.(98). In this article, the strain-energies

$$U(J_\mathrm{M}) = \frac{K}{50}\left(J_\mathrm{M}^5 + J_\mathrm{M}^{-5} + 2\right) \tag{102}$$

$$w(\mathrm{I}_{\overline{\mathbf{B}}}, \mathrm{II}_{\overline{\mathbf{B}}}) = \hat{\alpha}(\mathrm{I}_{\overline{\mathbf{B}}}^3 - 27) + c_{10}(\mathrm{I}_{\overline{\mathbf{B}}} - 3) + c_{01}(\mathrm{II}_{\overline{\mathbf{B}}}^{3/2} - 3\sqrt{3}) \tag{103}$$

with $\hat{\alpha} = 0.00367$ [MPa], $c_{01} = 0.1958$ [MPa] and $c_{10} = 0.1788$ [MPa], see [Hartmann and Neff, 2003], are applied. J_M is defined in Eq.(17). The first and second invariant of the unimodular left Cauchy-Green tensors have to be calculated,

$$\mathrm{I}_{\overline{\mathbf{B}}} = \mathrm{I}_{\overline{\mathbf{C}}} = \operatorname{tr}\overline{\mathbf{B}} = \overline{B} + 2\overline{B}_Q = (\lambda\lambda_Q^2)^{-2/3}(\lambda^2 + 2\lambda_Q^2) \tag{104}$$

$$\mathrm{II}_{\overline{\mathbf{B}}} = \mathrm{II}_{\overline{\mathbf{C}}} = \frac{1}{2}(\mathrm{I}_{\overline{\mathbf{B}}} - \operatorname{tr}\overline{\mathbf{B}}^2) = \operatorname{tr}(\overline{\mathbf{B}}^{-1}) = (\lambda\lambda_Q^2)^{2/3}(\lambda^{-2} + 2\lambda_Q^{-2}). \tag{105}$$

Obviously, the experimentally observed "linear" dependence of the temperature difference becomes obvious in a stress-stretch diagram, see Fig. 3. It has to be remarked, that the function $\hat{\varphi}(\Theta - \Theta_0)$ defined in Eq.(9) with

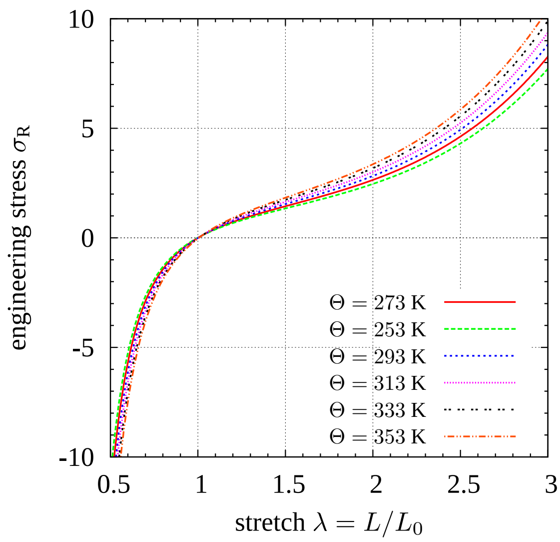


Figure 3: Simple tension for constant temperatures ($\Theta_0 = 273$ K). Representation of linear temperature dependence.

$\alpha = 2.06 \times 10^{-4}$ [K^{-1}], see [Heimes, 2005], has no essential influence in the range of applications, so that the behavior is close to a linear behavior.

From Eq.(64), expressed by the component representation (100) and (101), the two equations

$$\sigma = \frac{\rho_\mathrm{R}\Theta}{\varphi\Theta_0}U'(J/\varphi) + \frac{4\rho_\mathrm{R}\Theta}{3\lambda\lambda_Q^2\Theta_0}\left((w_1 + w_2\mathrm{I}_{\overline{\mathbf{B}}})(\overline{B} - \overline{B}_Q) - w_2(\overline{B}^2 - \overline{B}_Q^2)\right) \tag{106}$$

$$0 = \frac{\rho_\mathrm{R}\Theta}{\varphi\Theta_0}U'(J/\varphi) - \frac{2\rho_\mathrm{R}\Theta}{3\lambda\lambda_Q^2\Theta_0}\left((w_1 + w_2\mathrm{I}_{\overline{\mathbf{B}}})(\overline{B} - \overline{B}_Q) - w_2(\overline{B}^2 - \overline{B}_Q^2)\right) \tag{107}$$

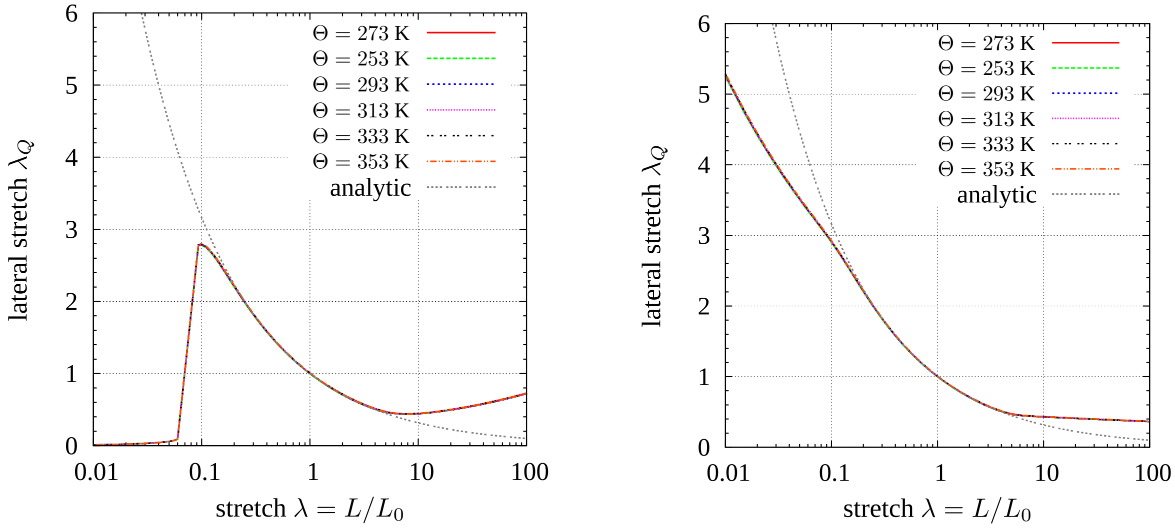
result. In other words, for given axial stretch λ and temperature Θ Eq.(107) has to be solved to obtain the lateral stretch λ_Q. In a second step the entire true stress (106) can be evaluated.

5.2 Problems with strain-energy functions

As mentioned in [Hartmann and Neff, 2003] the strain-energy part

$$U(J_\mathrm{M}) = \frac{K}{2}(J_\mathrm{M} - 1)^2 \tag{108}$$

yields in the case of uniaxial tensile tests to an increase of the lateral stretch in a certain amount of the lateral stretch (a specimen would become thicker in the tensile range). Accordingly, the investigation of the sensitivity of $J_\mathrm{M} = J/\varphi$ is of interest. In Fig. 4(a) the behavior of ansatz (108) is shown. However, there is no essential influence



(a) $U(J) = K/2(J-1)^2$ (b) $U(J) = K/50(J^5 + J^{-5} - 2)$

Figure 4: Lateral stretches for various strain-energy functions $U(J/\varphi)$

of the temperature-dependence introduced by $\hat{\varphi}(\Theta - \Theta_0)$. All curves are very close to each other. The proposal (102), however, does not show such a non-physical behavior, see Fig. 4(b), and the properties of the proposal in [Hartmann and Neff, 2003] are passed to the temperature-dependent case.

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